Math 10B with Professor Stankova
Worksheet, Discussion #22; Thursday, 4/12/2018
GSI name: Roy Zhao

1 Matrix Algebra

1.1 Concepts

1. A **matrix** is a $m \times n$ grid of numbers. This mean m rows and n columns. A **vector** can either be a **row vector** or **column vector**. A **row vector** is just a single row, so a $1 \times n$ matrix and a **column vector** is a column or a $m \times 1$ matrix. A **scalar** is just a number.

   We can add two matrices if they are of the same size. We add each entry separately. We can also multiply matrices by scalars.

   Given two matrices A, B that are of dimension $m \times n$ and $\ell \times k$, we can multiply them as AB if and only if $n = \ell$. So the number of columns in the first matrix must equal the number of rows in the second matrix. This means that sometimes we can compute AB but not BA. If you multiply a $m \times n$ matrix by a $n \times k$ matrix, the outcome is a $m \times k$ matrix.

   If two vectors have the same number of elements, then we can take the dot product of them. The **dot product** of the vectors $(a_1, a_2, \dots, a_n)$ and $(b_1, \dots, b_n)$ is a scalar given by $a_1b_1 + a_2b_2 + \cdots + a_nb_n$. We write it as $\vec{v} \circ \vec{w}$. The **norm** of a vector is given by $\sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$ and denoted by $|\vec{v}|$. Then $|a| = 0$ if and only if $a = 0$, the 0 vector. Let α be the angle between two vector $\vec{v}, \vec{w}$, then $\vec{v} \circ \vec{w} = |\vec{v}| \cdot |\vec{w}| \cos \alpha$.

   Cauchy-Schwarz inequality says that $|\vec{v} \circ \vec{w}| \leq |\vec{v}| \cdot |\vec{w}|$.

   For a $m \times n$ matrix A, the **transpose** A^T is the $n \times m$ matrix with all the elements flipped around.

1.2 Examples

2. Let $A = \begin{pmatrix} 3 & 5 & 6 \\ 1 & 2 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 0 \\ 3 & 3 \\ 0 & -2 \end{pmatrix}$. Calculate $A + 2B^T$ and AB.

Solution:

$$A + 2B^T = \begin{pmatrix} 1 & 11 & 6 \\ 1 & 8 & -2 \end{pmatrix}.$$

$$AB = \begin{pmatrix} 12 & 3 \\ 5 & 2 \end{pmatrix}$$

3. Find the angle between the two vector $v = (1, 3, 5, -2, 4, 3)$ and $w = (1, 1, 5, 2, 2, 1)$.

Solution: If α is the angle between them, then

$$\cos\alpha = \frac{v \circ w}{|v| \cdot |w|} = \frac{36}{\sqrt{64} \cdot \sqrt{36}} = \frac{36}{8 \cdot 6} = \frac{3}{4}.$$

Thus $\alpha = \arccos(3/4) \approx 0.7227$.

4. Suppose that $a + b + c + d = 1$. Prove that $a^2 + b^2 + c^2 + d^2 \geq \frac{1}{4}$.

Solution: Let $v = (a, b, c, d)$ and $w = (1, 1, 1, 1)$. Then Cauchy Schwarz tells us that $|v \circ w| \leq |v| \cdot |w|$ and

$$|v \circ w| = |(a \cdot 1) + (b \cdot 1) + (c \cdot 1) + (d \cdot 1)| = |a + b + c + d| = 1$$

while $|v| = \sqrt{a^2 + b^2 + c^2 + d^2}$ and $|w| = \sqrt{1^2 + 1^2 + 1^2 + 1^2} = \sqrt{4} = 2$. Thus $1 \leq 2\sqrt{a^2 + b^2 + c^2 + d^2}$ and squaring gives $a^2 + b^2 + c^2 + d^2 \geq \frac{1}{4}$ as required.

1.3 Problems

5. True **FALSE** If A is a matrix and v is a vector, then Av (assuming we can take such a product) is another vector.

Solution: If we multiply a 5×1 matrix with a 1×5 row vector, then the output is a 5×5 matrix that is not a vector. But, if v is a column vector, namely if v is 5×1, then the answer is true.

6. True **FALSE** If there are matrices such that $AB = M$ and we know the dimensions of M, then we know the dimensions of A and B.

Solution: A $(2 \times 2)(2 \times 2) = (2 \times 2)$ and $(2 \times 3)(3 \times 2) = (2 \times 2)$ so we need know the inside dimension as well.

7. Let $A = \begin{pmatrix} 2 & 3 & 3 \\ -1 & 0 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & -1 \\ -3 & -4 \\ 0 & 1 \end{pmatrix}$. Calculate AB and BA.

Solution:
$$AB = \begin{pmatrix} 1 & -11 \\ -5 & 5 \end{pmatrix} \qquad BA = \begin{pmatrix} 11 & 15 & 11 \\ -2 & -9 & -25 \\ -1 & 0 & 4 \end{pmatrix}.$$

8. Represent the system of equations $3x + 5y + 2z = 11, 8x - y = 0$ in matrix form as $A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = b$ with A being a matrix and b a vector.

Solution:
$$\begin{pmatrix} 3 & 5 & 2 \\ 8 & -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 11 \\ 0 \end{pmatrix}.$$

9. Let $v = (1, 2, 2, -1)$ and $w = (5, 3, -5, 3)$. Calculate $v \circ w$ and $|v|$.

Solution: $v \circ w = 1 \cdot 5 + 2 \cdot 3 + 2 \cdot (-5) + (-1) \cdot 3 = -2$. $|v| = \sqrt{1^2 + 2^2 + 2^2 + (-1)^2} = \sqrt{10}$.

10. Suppose that A is a matrix such that $A \begin{pmatrix} 1 \\ 1 \\ 11 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$. What are the dimensions of A? Come up with an example for A. Is the size unique? Is A unique?

Solution: Let A be a $m \times n$ matrix. Then $(m \times n) \cdot (3 \times 1) = (2 \times 1)$. Thus $m = 2$ and $n = 3$ and this size is unique. But there are many choices for A, one such choice is
$$A = \begin{pmatrix} 3 & 8 & -1 \\ 5 & 0 & 0 \end{pmatrix}$$

11. When is $|\vec{v} \circ \vec{w}| = |\vec{v}| \cdot |\vec{w}|$? (Hint: What is α?)

Solution: We know that $|\vec{v} \circ \vec{w}| = ||v| \cdot |w| \cdot \cos\alpha| = |v| \cdot |w| \cdot |\cos\alpha|$. Thus $|\cos\alpha| = 1$ and hence $\alpha = 0, \pi$. Therefore, the vectors must on the same line.

12. Find a 2×2 matrix A with no 0's such that $A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Solution: One choice is $\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$.

13. Find x, y such that

$$\begin{pmatrix} -3 & 2 \\ y & x \end{pmatrix} \begin{pmatrix} -2 & 5 \\ x & y \end{pmatrix} = \begin{pmatrix} y & -7 \\ -7 & 16 \end{pmatrix}$$

Solution: Multiplying gives

$$\begin{pmatrix} 6+2x & 2y-15 \\ x^2-2y & xy+5y \end{pmatrix} = \begin{pmatrix} y & -7 \\ -7 & 16 \end{pmatrix}.$$

Thus $2y - 15 = -7$ and hence $2y = 8$ or $y = 4$. Then we have that $2x + 6 = y = 4$ so $2x = 4 - 6 = -2$ and $x = -1$.

2 Determinants and Inverses

2.1 Concepts

14. The **determinant** is defined only for square matrices. It is a scalar. The determinant for a 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the number $ad - bc$.

The **inverse** of a matrix is defined only for square matrices. The inverse of A is a matrix B such that $AB = BA = I$, the identity matrix with 1s on the diagonal and 0 everywhere else. The inverse is always unique if it exists. The inverse of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the matrix $B = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. The inverse of a square matrix exists if and only if the determinant is nonzero.

We can use the inverse to easily solve equations of the form $Ax = b$ where x, b are vectors and A is a matrix because we can write $x = A^{-1}b$ if A is invertible. This always has a **unique** solution if A is invertible. If A is not invertible, this is 0 solutions or ∞ solutions.

2.2 Examples

15. Find x, y such that $2x + 3y = 4$ and $x + y = 1$.

Solution: Let $A = \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}$, then $A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ and hence

$$\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1} \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \frac{1}{2 \cdot 1 - 3 \cdot 1} \begin{pmatrix} 1 & -3 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

2.3 Problems

16. **TRUE** False We can take determinants of 3×3 matrices but just haven't learned it yet.
17. True **FALSE** We can take determinants of 2×3 matrices but just haven't learned it yet.
18. True **FALSE** If A is a noninvertible square matrix, then $Ax = b$ may still have a unique solution.

Solution: If A is not invertible, then $Ax = b$ has 0 or ∞ solutions.

19. True **FALSE** If $\det(A) = 0$, then $Ax = b$ has no solutions.

Solution: It is possible for it to have ∞ solutions.

20. Give a 2×2 matrix with determinant equal to 5. Is it unique?

Solution: This is not unique. One solution is $\begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix}$.

21. Find the inverses for the following matrices:

$$\begin{pmatrix} 3 & 5 \\ -4 & -8 \end{pmatrix} \qquad \begin{pmatrix} 2 & 5 \\ 3 & 4 \end{pmatrix} \qquad \begin{pmatrix} 1 & 5 \\ -1 & -8 \end{pmatrix}$$

Solution:

$$\begin{pmatrix} 2 & 5/4 \\ -1 & -3/4 \end{pmatrix} \qquad \begin{pmatrix} -4/7 & 5/7 \\ 3/7 & -2/7 \end{pmatrix} \qquad \begin{pmatrix} 8/3 & 5/3 \\ -1/3 & -1/3 \end{pmatrix}.$$

22. Find a matrix X such that $\begin{pmatrix} 5 & 13 \\ 3 & 8 \end{pmatrix} X = \begin{pmatrix} 1 & 4 & 1 \\ -1 & 2 & 1 \end{pmatrix}$.

Solution:

$$X = \frac{1}{5 \cdot 8 - 13 \cdot 3} \begin{pmatrix} 8 & -13 \\ -3 & 8 \end{pmatrix} \begin{pmatrix} 1 & 4 & 1 \\ -1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 21 & 6 & -5 \\ -8 & -2 & 2 \end{pmatrix}$$